5) Figure 2 indicates very good comparisons with previous work[2] (fuel minimization for specified energy of escape). Even for large radial distance r, the control angle $\psi = \tan^{-1}[\hat{a}_r/\hat{a}_\theta] = \tan^{-1}[\eta]$ is very close to the direction of optimal increase of energy.

6) Good performance of the guidance scheme utilized in the paper depends upon J_1 and J_2 being close together. Reference 9 indicates that the guidance equations applied to J_1 or J_2 have negligible differences. A future paper by the authors will discuss the problem of decreasing thrust magnitude with altitude and will develop a compensating control scheme.

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Optimum Estimation and Coordinate Conversion for Radio Navigation

W. D. BARD* AND T. E. PERFITT†

Lear Siegler Inc., Instrument Division, Grand Rapids, Mich.

Radio navigation aids such as OMEGA and LORAN can provide many hyperbolic lines of position (LOP's) that, in general, do not have a common point of intersection. This paper describes a technique that uses the extra LOP's which are linearly independent to provide the most probable receiver location in geodetic coordinates. The approximate nonlinear algebraic equations that maximize the a posteriori density function are given and a two-dimensional Newton-Raphson scheme presented for their solution. The covariance of the geodetic estimation error is shown to be the inverse of the Jacobian matrix associated with the Newton-Raphson scheme for small measurement noise. This matrix is used to develop accuracy contour plots for the four OMEGA transmitting sites that are presently on the air. An improvement in OMEGA accuracy of about one-half mile can be expected when this four station technique is used to replace a conventional three station coordinate conversion scheme.

Nomenclature

ϕ,λ	=	receiver latitude and longitude, respectively
$\hat{\phi},\hat{\lambda}$	=	estimates of ϕ and λ, respectively
t_i	=	time-of-arrival of signal from the ith transmitter
d_i	=	geodetic distance between receiver and ith transmitter
c	=	speed of light
n	=	number of transmitters
$\mathbf{D}$	=	$(n-1)\times 1$ vector of distance differences
$\mathbf{T}$	=	$(n-1)\times 1$ vector of time differences
$\boldsymbol{\varepsilon}$	=	$(n-1)\times 1$ vector of time difference errors
$\mathbf{S}$	=	$(n-1)\times 1$ vector equal to $\mathbf{T}-(1/c)\mathbf{D}$
$\mathbf{\Phi}$	=	$(n-1)\times(n-1)$ covariance matrix of time difference errors
σ_i^2	=	variance of time of arrival measurement from the ith transmitter
θ_i	=	azimuth angle of transmitter i measured at receiver
R	=	earth radius
$C_n(2)$	=	number of combinations of n things taken two at a time

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* Mathematical Analyst, Analytical Engineering Department.

† Manager, Systems Staff.

Introduction

THE expansion of the OMEGA hyperbolic navigation system to include eight ground based radio transmitters poses questions of the following type: 1) if redundant lines of position (LOP's) are available with many intersecting points, how should the information be combined to provide the most probable position of the receiver? 2) neglecting lane ambiguity, should all the LOP's or even possibly all the intersection points of the LOP's be used in calculating the most probable receiver location? 3) is it necessary to coordinate convert all the LOP information to geodetic coordinates before statistically averaging or can the averaging and coordinate conversion be done simultaneously?

This paper answers these questions by describing a technique that uses the minimum LOP information necessary to obtain a most probable estimate of receiver location. The estimate is obtained in the desired geodetic coordinate system directly from hyperbolic time difference information, i.e., the estimation and coordinate conversion are performed simultaneously.

An error analysis was reported by Marchand[1] in 1964 that obtained the error distribution of the most probable estimator without actually obtaining explicit equations for the estimator. His analysis, however, assumes that the errors in the $C_n(2)$ hyperbolic lines of position available from n transmitters are statistically independent. We show here that only $n-1$ of the errors are even linearly independent and that these are in general strongly correlated. Our error analysis corrects this deficiency and displays the actual accuracy of the estimator in terms of contour plots.

Development of Estimator Equations

For this analysis, we assume that 1) there are $C_n(2)$ time of arrival difference measurements available from n radio transmitters, 2) predictable errors due to changing propagation velocities are removed, 3) the mean of the time difference measurement errors are zero, i.e., the corrected time differences averaged over many months of operation provide LOP's that pass through the actual position of the receiver, and 4) lane ambiguities due to periodicity of the r.f. waveform are removed.

The most probable receiver location after $C_n(2)$ time difference measurements are taken is determined by maximizing the conditional joint probability density function of the geodetic receiver coordinates.[2] In notational form, the most probable receiver location, $(\hat{\lambda},\hat{\phi})$, maximizes the a posteriori density function

$$p(\lambda,\phi|t_1-t_2,t_1-t_3,\ldots,t_2-t_3,\ldots,t_{n-1}-t_n)$$

over all λ and ϕ.

In formulating the a posteriori density function, however, not all $C_n(2)$ time difference measurements are required. Only $n-1$ are linearly independent and the remaining ones are deterministically related to these. In matrix form

$$\begin{bmatrix} t_2-t_1 \\ t_3-t_1 \\ \cdot \\ \cdot \\ t_n-t_1 \\ t_3-t_2 \\ \cdot \\ \cdot \\ t_n-t_{n-1} \end{bmatrix} = M^* \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \cdot \\ \cdot \\ t_n \end{bmatrix} \quad (1)$$

where t_i is the time of arrival of the ith transmitted signal and M^* is a $C_n(2)\times n$ matrix of zeros except for a single plus and minus one in each row. The rank of M^* is no greater than $n-1$ since the sum of any $n-1$ columns is equal to the negative of the remaining column. Thus, there is a maximum of $n-1$ linearly independent time difference measurements. There are, however, $n-1$ independent measurements available. Since it is unimportant which set is used, it is convenient to select

$$\begin{bmatrix} t_2-t_1 \\ t_3-t_1 \\ \cdot \\ \cdot \\ \cdot \\ t_n-t_1 \end{bmatrix} \equiv \mathbf{T}$$

as the linearly independent set.

Since the time differences between all stations can be computed by simply subtracting the appropriate measured entries of **T**, the $C_n(2)$ time differences convey no more information about the receiver location than do the entries of **T**. It therefore follows that

$$p(\lambda,\phi|t_1-t_2,t_1-t_3,\ldots,t_2-t_3,\ldots,t_{n-1}-t_n) = p(\lambda,\phi|t_2-t_1,t_3-t_1,\ldots,t_n-t_1) \quad (2)$$

Applying Bayes' theorem to the right-hand side of (2) provides

$$p(\lambda,\phi|t_2-t_1,t_3-t_1,\ldots,t_n-t_1) = \frac{p(t_2-t_1,t_3-t_1,\ldots,t_n-t_1|\lambda,\phi)p(\lambda,\phi)}{p(t_2-t_1,t_3-t_1,\ldots,t_n-t_1)} \quad (3)$$

The denominator of (3) is constant with respect to ϕ and λ and may be ignored in the maximizing process. Likewise, the a priori distribution on the receiver coordinates, $p(\lambda,\phi)$, is very "broad" and essentially constant near the maximum of (3) and may also be ignored. Thus, the most probable receiver location is the pair $(\hat{\lambda},\hat{\phi})$ that maximizes

$$p(t_2-t_1,t_3-t_1,\ldots,t_n-t_1|\lambda,\phi) \quad (4)$$

over all ϕ and λ. In words, (4) is the probability density of the $n-1$ time difference measurements conditioned on the fact that the receiver is located at the point (λ,ϕ).

The actual form of (4) is generated by assuming that the arrival time errors are statistically independent from one station to another and have a gaussian distribution. Independence appears reasonable, for example, for the OMEGA navigation system since the primary source of error is the inaccurate diurnal phase correction applied to the arrival time of each station. The corrections are based on (among other things) the geography and path length between the receiver and transmitter. Since the paths are normally quite dissimilar, the independence assumption is reasonable. Experimental results[3] indicate that the predicted time of arrival errors for OMEGA are nearly gaussian.

Letting e_i represent the error in the time difference measurement, $t_{i+1}-t_1$, we define

$$E\left\{\begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_{n-1} \end{bmatrix} [e_1 e_2,\ldots,e_{n-1}]\right\} = \sigma_1^2 \begin{bmatrix} 1+\left(\frac{\sigma_2}{\sigma_1}\right)^2 & 1 & 1 & \cdots & 1 \\ 1 & 1+\left(\frac{\sigma_3}{\sigma_1}\right)^2 & 1 & \cdots & 1 \\ 1 & 1 & 1+\left(\frac{\sigma_4}{\sigma_1}\right)^2 & \cdots & 1 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ 1 & 1 & & \cdots & 1+\left(\frac{\sigma_n}{\sigma_1}\right)^2 \end{bmatrix} \equiv \Phi \quad (5)$$

where σ_i^2 is the variance of the ith time of arrival error. We assume that each σ_i is either known or measured.

The form of (4) can now be written as

$$p(t_2-t_1,t_3-t_1,\ldots,t_n-t_1|\lambda,\phi) = \frac{1}{[(2\pi)^{n-1}\det\Phi]^{1/2}}\exp -\tfrac{1}{2}S'\Phi^{-1}S \quad (6)$$

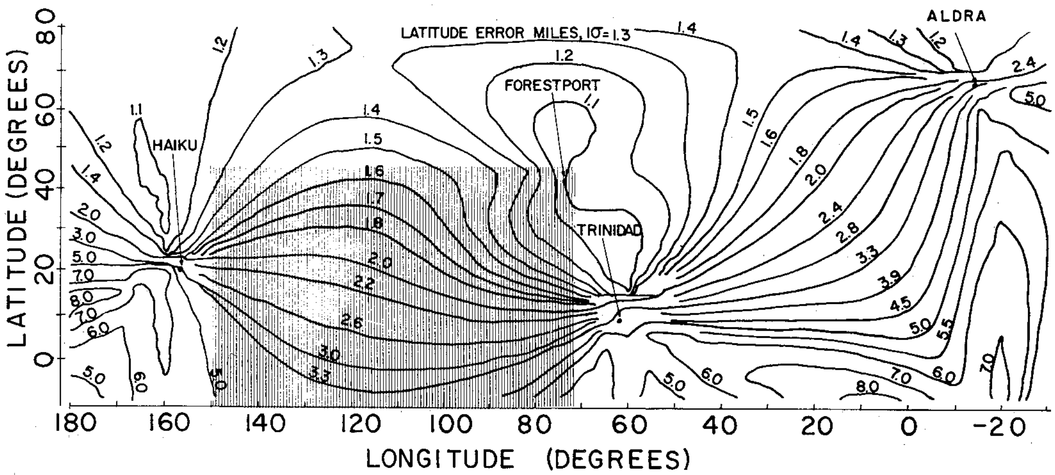

Fig. 1 Latitude accuracy plots (four stations).

where

$$S = \begin{bmatrix} t_2 - t_1 - (1/c)[d_2(\phi,\lambda) - d_1(\phi,\lambda)] \\ t_3 - t_1 - (1/c)[d_3(\phi,\lambda) - d_1(\phi,\lambda)] \\ \cdot \\ \cdot \\ \cdot \\ t_n - t_1 - (1/c)[d_n(\phi,\lambda) - d_1(\phi,\lambda)] \end{bmatrix} \tag{7}$$

Here, c is the velocity of light and $d_i(\phi,\lambda)$ is the geodetic distance between transmitter i and a receiver located at the point (λ,ϕ).

Maximizing (6) is the same as minimizing the quadratic exponent

$$V(\lambda,\phi,T) = \tfrac{1}{2}S'\Phi^{-1}S \tag{8}$$

over all ϕ and λ. Two necessary conditions for (8) to be minimum are

$$\delta V/\delta\lambda = V_\lambda = S'_\lambda\Phi^{-1}S = 0 \tag{9}$$

and

$$\delta V/\delta\phi = V_\phi = S'_\phi\Phi^{-1}S = 0 \tag{10}$$

By virtue of the distances d, the vectors S_ϕ, S_λ, and S are all nonlinear functions of the receiver location λ and ϕ. The roots of the two nonlinear algebraic equations (9) and (10) represent the most probable receiver location $(\hat{\lambda},\hat{\phi})$. Thus, if $n - 1$ measured, linearly independent time differences are substituted into (9) and (10) and these equations are solved, the results provide the most probable receiver location $(\hat{\lambda},\hat{\phi})$ expressed in geodetic coordinates.

Iteration Technique

The practical solution to the algebraic equations (9) and (10) requires an iterative scheme that converges rapidly and is simple to implement. For these equations, the two-dimensional Newton-Raphson technique satisfies these requirements plus has one other important feature. It is established in Appendix B that, for small measurement noise, the inverse Jacobian matrix of the Newton-Raphson technique is also the covariance of the geodetic estimation error. It is especially important to have this covariance matrix available if the estimates $\hat{\lambda}$ and $\hat{\phi}$ are to be subsequently combined with other independent navigation aids. The matrix may also be used to display error bounds on both $\hat{\phi}$ and $\hat{\lambda}$.

Expressed in terms of the partial derivatives of V, the Newton-Raphson scheme is

$$\begin{bmatrix} \hat{\lambda} \\ \hat{\phi} \end{bmatrix}_{k+1} = \begin{bmatrix} \hat{\lambda} \\ \hat{\phi} \end{bmatrix}_k - J^{-1} \begin{bmatrix} V_\lambda \\ V_\phi \end{bmatrix}_k \tag{11}$$

where

$$J = \begin{bmatrix} V_{\lambda\lambda} & V_{\phi\lambda} \\ V_{\lambda\phi} & V_{\phi\phi} \end{bmatrix}_k$$

and

$$\begin{bmatrix} \hat{\lambda} \\ \hat{\phi} \end{bmatrix}_0$$

is an initial position estimate. As developed in Appendix A, these equations can be expressed in terms of the following known or approximately known quantities: 1) the latitude and longitude of the initial position estimate, 2) the geodetic distances (d_i) between the transmitter and the initial position estimate, 3) the azimuth angles (θ_i), and 4) the observed time differences (**T**).

In detail, the equations are

$$V_\lambda = S'_\lambda\Phi^{-1}S,\ V_\phi = S'_\phi\phi^{-1}S,\ V_{\lambda\lambda} = S'_\lambda\Phi^{-1}S_\lambda$$
$$V_{\lambda\phi} = S'_\lambda\Phi^{-1}S_\phi,\ V_{\phi\phi} = S'_\phi\Phi^{-1}S_\phi \tag{12}$$

Table 1 Mean absolute error convergence of 50 cases for 10 locations

True position		Initial bias, ft		First iteration mean error, ft		Second iteration mean error, ft	
Lon	Lat	Lon	Lat	Lon	Lat	Lon	Lat
160°	45°	10712	7980	22.84	28.86	0.01	0.03
140°	45°	7396	8634	32.59	25.36	0.05	0.05
120°	45°	7148	9923	29.77	12.37	0.03	0.02
100°	45°	5806	8421	19.26	10.10	0.02	0.04
80°	45°	5896	7097	33.56	15.86	0.08	0.08
60°	45°	7760	8140	16.40	13.07	0.03	0.03
40°	45°	9011	8793	15.92	44.05	0.01	0.10
20°	45°	10553	12010	19.33	60.37	0.02	0.20
0	45°	14691	14108	38.25	74.22	0.10	0.31
−20°	45°	29610	30592	202.40	218.61	1.93	2.29

Table 2 Comparison of predicted and simulation standard deviations

True position		Predicted from Jacobian, ft		Computed from simulation, ft	
Lon	Lat	Lon	Lat	Lon	Lat
160°	45°	11900	6600	11700	6500
140°	45°	8200	9000	8400	8200
120°	45°	6500	10000	6400	8000
100°	45°	6200	9400	5300	8800
80°	45°	7000	7300	6600	7900
60°	45°	7800	7000	8200	7100
40°	45°	9300	8700	8900	8500
20°	45°	11700	12200	11700	12000
0	45°	18500	18700	18800	17500
−20°	45°	32500	35200	34600	36500

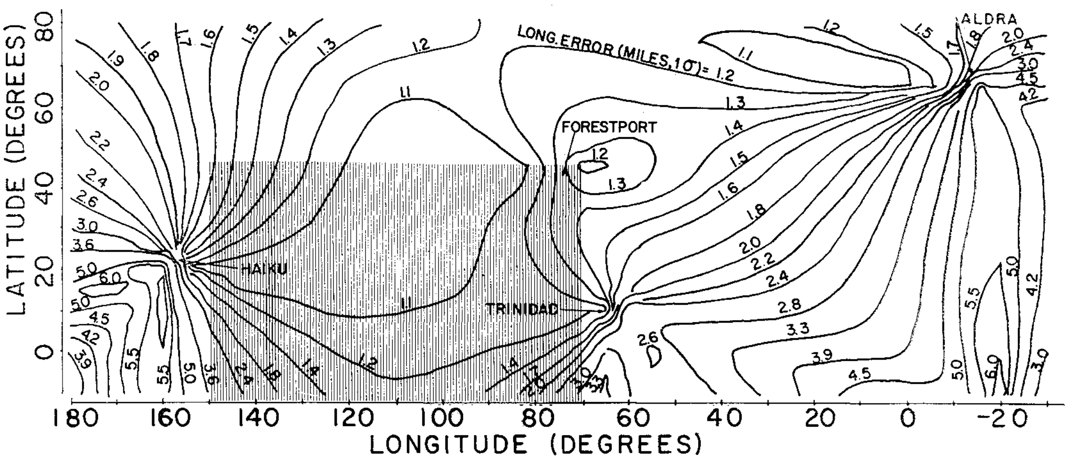

Fig. 2 Longitude accuracy plots (four stations).

The ith component of the partial derivatives of S is

$$\partial s_i/\partial\lambda = -(R/c)\cos\phi(\cos\theta_{i+1} - \cos\theta_1)$$
$$\partial s_i/\partial\phi = (R/c)(\sin_{i+1} - \sin\theta_1) \qquad (13)$$

It is also shown in Appendix A that the Jacobian matrix J is nonsingular if operation is restricted away from the earth's poles and at least 3 stations have different azimuth angles. Thus, for all cases of practical interest, the Newton-Raphson scheme converges.[4]

Simulation Results

To illustrate the speed of convergence of this technique, time of arrival measurements from four OMEGA stations at Aldra, Trinidad, Forestport, and Haiku were simulated. Gaussian random numbers with a standard deviation of 10 centicycles (cecs) (or equivalently 1.6 naut miles) were used to simulate the measurement errors. This standard deviation was selected on the basis of experimental results reported in Ref. 3.

For each of several positions within the converge area, the Newton-Raphson procedure was simulated using a random initial position estimate. Table 1 summarizes the results of 50 such cases at each of ten locations, showing mean absolute errors for each iteration and for the final estimate. Table 2 shows the standard deviations of the geodetic estimation errors as computed from the 50 cases and as predicted by the Jacobian matrix entries. It is noteworthy that convergence is somewhat slower for those locations lying in poorer regions of the coverage area. Sufficient accuracy is always achieved, however, after two iterations are performed.

Estimation Accuracy

To demonstrate the desirability of this approach, the estimation accuracy as determined by the entries of the Jacobian matrix are plotted as accuracy contour maps in Figs. 1 and 2 for all four stations. Again, time of arrival errors of 10 cecs are assumed. Figures 3 and 4 show the standard deviations resulting from the use of only three stations, Forestport, Trinidad, and Haiku, with the shaded areas indicating "good" coverage areas for these three stations. Figure 5 indicates the improvement found by using four stations instead of three. The statistic that is compared in Fig. 5 is the square root of the sum of the latitude and longitude variances.

The most significant point here is the improvement displayed in Fig. 5. In areas that are considered to be good by a conventional coordinate conversion scheme, the technique described in this paper (using four stations) provides a conservative one-half mile average increase in navigational accuracy.

Conclusions

The technique presented in this paper provides a means of simultaneously combining redundant hyperbolic lines of position and performing coordinate conversion. The technique converges rapidly and is simple enough to be implemented on a small airborne digital computer. The improvement in accuracy that this technique offers over conventional coordinate conversion using only three stations is significant for the OMEGA navigation system. As a by-product of the technique, the covariance of the geodetic estimation errors is available as part of the iteration equations.

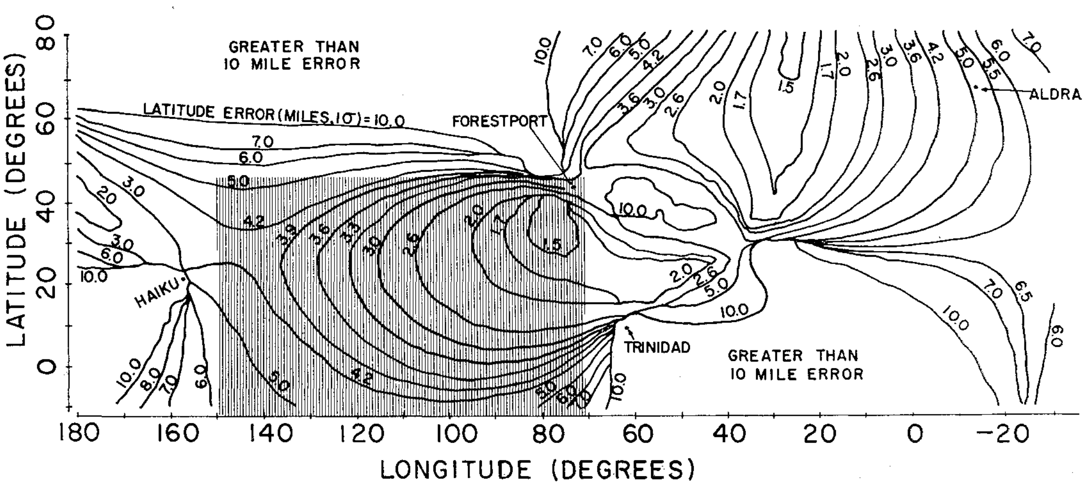

Fig. 3 Latitude accuracy plots (three stations).

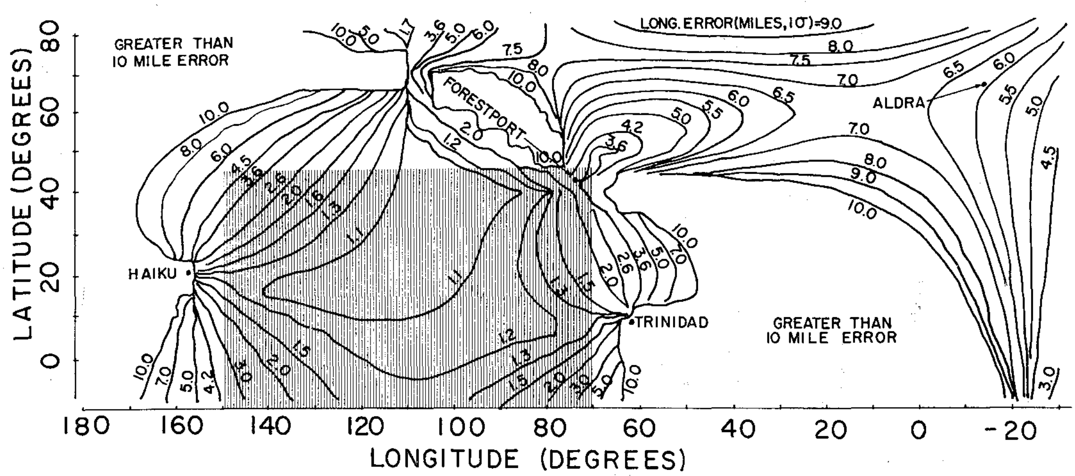

Fig. 4 Longitude accuracy plots (three stations).

Appendix A: Equation Development for the Newton-Raphson Scheme

The equation development and the geometrical approximations used for the Newton-Raphson scheme follow. The partial derivatives of V of Eq. (8) are given by

$$\begin{aligned} V_\lambda &= S'_\lambda\Phi^{-1}S,\ V_\phi = S'_\phi\Phi^{-1}S \\ V_{\lambda\lambda} &= S'_\lambda\,\Phi^{-1}S_\lambda + S'_{\lambda\lambda}\,\Phi^{-1}S \\ V_{\lambda\phi} &= S'_\lambda\Phi^{-1}S_\phi + S'_{\lambda\phi}\phi^{-1}S \\ V_{\phi\phi} &= S'_\phi\Phi^{-1}S_\phi + S'_{\phi\phi}\Phi^{-1}S \end{aligned} \tag{A1}$$

Hence, the problem is reduced to expressing the vectors $S_\lambda, S_\phi, S_{\lambda\lambda}, S_{\lambda\phi}$, and $S_{\phi\phi}$ in simple terms of known or approximately known quantities.

The ith component of S is

$$s_i = [t_{i+1} - t_1 - (1/c)(d_{i+1} - d_1)]$$

Now consider the rectangular coordinate system of Fig. 6 defined at the present position λ and ϕ.

Figure 6 provides

$$\partial d_i/\partial x = \lim_{\Delta x\to 0} \Delta d/\Delta x = -\cos\theta_i$$

Likewise

$$\partial d_i/\partial y = \lim_{\Delta y\to 0} \Delta d/\Delta y = -\sin\theta_i$$

For large transmitter to receiver distances, θ_i is essentially constant in the region of interest. Hence

$$\partial^2 d_i/\partial x^2 = (\sin\theta_i)\partial\theta_i/\partial x = 0 \tag{A2}$$

and

$$\partial^2 d_i/\partial x\partial y = \partial^2 d_i/\partial y^2 = 0 \tag{A3}$$

Thus, the terms involving $S_{\lambda\lambda}$, $S_{\lambda\phi}$, and $S_{\phi\phi}$ in (A1) can be neglected.

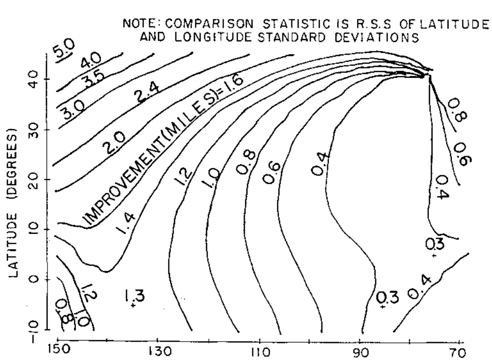

Fig. 5 Accuracy improvement of four vs three stations.

Considering the earth to be a perfect sphere for evaluating the partial derivatives provides

$$\partial x/\partial\lambda = -R\cos\phi \tag{A4}$$

and

$$\partial y/\partial\phi = R \tag{A5}$$

Thus, the necessary partial derivatives of the vector S are expressable as a simple function of the approximately known latitude angle and transmitter azimuth angles. In detail

$$\partial s_i/\partial\lambda = -(R/c)\cos\phi(\cos\theta_{i+1} - \cos\theta_1) \tag{A6}$$

and

$$\partial s_i/\partial\phi = (R/c)(\sin\theta_{i+1} - \sin\theta_1) \tag{A7}$$

Jacobian Inverse

The implementation of this scheme requires the inverse of the Jacobian matrix given in Eq. (11). It is asserted that the inverse of this matrix always exists at $(\hat{\lambda},\hat{\phi})$ if at least three stations have different azimuth angles and operation is restricted away from the earth poles. The proof of this statement follows.

From Eqs. (11) and (12) J is equal to

$$\begin{bmatrix} S'_\lambda & \Phi^{-1}S_\lambda & S'_\phi & \Phi^{-1}S_\lambda \\ S'_\lambda & \Phi^{-1}S_\phi & S_\phi & \Phi^{-1}S_\phi \end{bmatrix} = \begin{bmatrix} S'_\lambda \\ S'_\phi \end{bmatrix}\Phi^{-1}[S_\lambda : S_\phi] \tag{A8}$$

For nonzero σ_i, Φ and Φ^{-1} are positive definite. The Jacobian J is therefore positive definite if the rank of $[S_\lambda : S_\phi]$ is two. But assume

$$[S_\lambda : S_\phi] = \begin{bmatrix} -(R/c)\cos\phi(\cos\theta_2 - \cos\theta_1) & (R/c)(\sin\theta_2 - \sin\theta_1) \\ \vdots & \vdots \\ -(R/c)\cos\phi(\cos\theta_n - \cos\theta_1) & (R/c)(\sin\theta_n - \sin\theta_1) \end{bmatrix}$$

is of rank one, θ_1, θ_2, and θ_3 are the different angles, and opera-

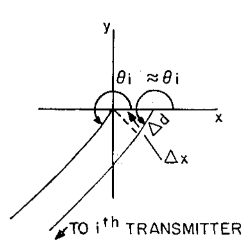

Fig. 6 Local coordinate system.

tion is restricted away from the poles, i.e., $\cos\phi \neq 0$. Then for some constant α, $0 \leq \alpha < 2\pi$

$$\sin\alpha(\cos\theta_{i+1} - \cos\theta_1) = \cos\alpha(\sin\theta_{i+1} - \sin\theta_1)$$

for $i = 1,2$. In other words

$$\sin\alpha\, \cos\theta_{i+1} - \cos\alpha\, \sin\theta_{i+1} = \sin\alpha\, \cos\theta_1 - \cos\alpha\, \sin\theta_1$$

or

$$\sin(\alpha - \theta_2) = \sin(\alpha - \theta_3) = \sin(\alpha - \theta_1)$$

Since the sine function is double valued in the interval $[\alpha,\alpha + 2\pi]$, two of the azimuth angles must be equal. But, this is a contradiction. Hence, if operation is restricted away from the earth poles and at least three stations have different azimuth angles, then $J(\hat{\phi},\hat{\lambda})$ is nonsingular.

Appendix B: Error Analysis

It is asserted that for small measurement noise, the covariance of the geodetic estimation error is the inverse of the Jacobian matrix in Eq. (11). To prove this, let $\boldsymbol{\varepsilon}$ be the $(n - 1) \times 1$ vector of time difference errors with ith component

$$\epsilon_i = t_{i+1} - t_1 - (1/c)(d_{i+1} - d_1)\quad i = 1,2,3, \ldots ,n - 1$$

From Eqs. (9) and (10), the estimates $\hat{\lambda}(\boldsymbol{\varepsilon})$ and $\hat{\phi}(\boldsymbol{\varepsilon})$ must satisfy

$$V_\lambda[\hat{\lambda}(\boldsymbol{\varepsilon}),\hat{\phi}(\boldsymbol{\varepsilon}),\boldsymbol{\varepsilon}] \equiv 0$$

and

$$V_\phi[\hat{\lambda}(\boldsymbol{\varepsilon}),\hat{\phi}(\boldsymbol{\varepsilon}),\boldsymbol{\varepsilon}] \equiv 0$$

for all $\boldsymbol{\varepsilon}$. Differentiating these equations with respect to the first component of $\boldsymbol{\varepsilon}$ provides

$$V_{\lambda\lambda}\partial\hat{\lambda}/\partial\mathcal{E}_1 + V_{\lambda\phi}\partial\hat{\phi}/\partial\mathcal{E}_1 + \partial V_\lambda/\partial\mathcal{E}_1 = 0 \tag{B1}$$

and

$$V_{\phi\lambda}\frac{\partial\hat{\lambda}}{\partial\mathcal{E}_1} + V_{\phi\phi}\frac{\partial\hat{\phi}}{\partial\mathcal{E}_1} + \frac{\partial V_\phi}{\partial\mathcal{E}_1} = 0 \tag{B2}$$

Repeating the differentiation with respect to the remaining components of $\boldsymbol{\varepsilon}$ provides the matrix equation

$$\begin{bmatrix} V_{\lambda\lambda} & V_{\lambda\phi} \\ V_{\phi\lambda} & V_{\phi\phi} \end{bmatrix}\begin{bmatrix} \hat{\lambda}'_\mathcal{E} \\ \hat{\phi}'_\mathcal{E} \end{bmatrix} = -\begin{bmatrix} V'_{\lambda\mathcal{E}} \\ V'_{\phi\mathcal{E}} \end{bmatrix} \tag{B3}$$

where

$$\hat{\lambda}'_\epsilon = \begin{bmatrix}\frac{\partial\hat{\lambda}}{\partial\mathcal{E}_1}\frac{\partial\lambda}{\partial\mathcal{E}_2}\cdots\frac{\partial\hat{\lambda}}{\partial\mathcal{E}_{n-1}}\end{bmatrix}$$

$$\hat{\phi}'_\epsilon = \begin{bmatrix}\frac{\partial\hat{\phi}}{\partial\mathcal{E}_1}\frac{\partial\hat{\phi}}{\partial\mathcal{E}_2}\cdots\frac{\partial\hat{\phi}}{\partial\mathcal{E}_{n-1}}\end{bmatrix}$$

$$V'_{\lambda\epsilon} = \begin{bmatrix}\frac{\partial V_\lambda}{\partial\mathcal{E}_1}\frac{\partial V_\lambda}{\partial\mathcal{E}_2}\cdots\frac{\partial V_\lambda}{\partial\mathcal{E}_{n-1}}\end{bmatrix}$$

$$V'_{\phi\epsilon} = \begin{bmatrix}\frac{\partial V_\phi}{\partial\mathcal{E}_1}\frac{\partial V_\phi}{\partial\mathcal{E}_2}\cdots\frac{\partial V_\phi}{\partial\mathcal{E}_{n-1}}\end{bmatrix}$$

From Eq. (11),

$$J = \begin{bmatrix} V_{\lambda\lambda} & V_{\phi\lambda} \\ V_{\lambda\phi} & V_{\phi\phi} \end{bmatrix}$$

Since J is symmetric and nonsingular for all cases of interest,

$$\begin{bmatrix} \hat{\lambda}'_\mathcal{E} \\ \hat{\phi}'_\mathcal{E} \end{bmatrix} = -J^{-1}\begin{bmatrix} V'_{\lambda\mathcal{E}} \\ V'_{\phi\mathcal{E}} \end{bmatrix} \tag{B4}$$

Now, consider a Taylor series expansion of $\hat{\lambda}$ and $\hat{\phi}$ about $\boldsymbol{\varepsilon} = 0$.

$$\hat{\lambda}(\boldsymbol{\varepsilon}) - \hat{\lambda}(0) = \sum_{i=1}^{n-1}\frac{\partial\lambda}{\partial\mathcal{E}_i}\bigg|_{\mathcal{E}=0}\epsilon_i + 0(\epsilon_i^2) \tag{B5}$$

and

$$\hat{\phi}(\boldsymbol{\varepsilon}) - \hat{\phi}(0) = \sum_{i=1}^{n-1}\frac{\partial\hat{\phi}}{\partial\mathcal{E}_i}\bigg|_{\mathcal{E}=0}\mathcal{E}_i + 0(\mathcal{E}_i^2) \tag{B6}$$

Neglecting $0(\mathcal{E}_i^2)$ terms for small noise, we have

$$\begin{bmatrix} \hat{\lambda}(\boldsymbol{\varepsilon}) - \hat{\lambda}(0) \\ \hat{\phi}(\boldsymbol{\varepsilon}) - \hat{\phi}(0) \end{bmatrix} \equiv \begin{bmatrix} \Delta\lambda \\ \Delta\phi \end{bmatrix} = \begin{bmatrix} \hat{\lambda}'_\mathcal{E} \\ \hat{\phi}'_\mathcal{E} \end{bmatrix}\boldsymbol{\varepsilon}$$

The expected value of $\Delta\lambda$ and $\Delta\phi$ is zero for zero mean measurement noise. The covariance of $\Delta\lambda$ and $\Delta\phi$ is

$$E\left\{\begin{bmatrix} \Delta\lambda \\ \Delta\phi \end{bmatrix}[\Delta\lambda\ \ \Delta\phi]\right\} = \begin{bmatrix} \hat{\lambda}'_\mathcal{E} \\ \hat{\phi}'_\mathcal{E} \end{bmatrix} E[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'][\hat{\lambda}_\epsilon\hat{\phi}_\epsilon]\bigg|_{\mathcal{E}=0} \tag{B7}$$

where E is the expected value operator. However

$$E[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'] = \Phi \tag{B8}$$

Substituting (B8) and (B4) into (B7) provides

$$E\begin{bmatrix} \Delta\lambda^2 & \Delta\lambda\Delta\phi \\ \Delta\lambda\Delta\phi & \Delta\phi^2 \end{bmatrix} = J^{-1}\begin{bmatrix} V'_{\lambda\mathcal{E}} \\ V'_{\phi\mathcal{E}} \end{bmatrix}\Phi[V_{\lambda\mathcal{E}}V_{\phi\mathcal{E}}]J^{-1} \tag{B9}$$

where the right-hand side is evaluated at $\boldsymbol{\varepsilon} = 0$. Since

$$V_\lambda = S'_\lambda\Phi^{-1}S$$

and

$$V_\phi = S'_\phi\Phi^{-1}S$$

it follows that

$$V_{\lambda\mathcal{E}_1}\big|_{\mathcal{E}=0} = \left\{S'_{\lambda\mathcal{E}_1}\Phi^{-1}S + S'_\lambda\Phi^{-1}\begin{bmatrix}1\\0\\0\\0\\ \cdot\\ \cdot\\ 0\end{bmatrix}\right\}_{\mathcal{E}=0} = S'_\lambda\Phi^{-1}\begin{bmatrix}1\\0\\0\\ \\ \cdot\\ \cdot\\ 0\end{bmatrix}_{\mathcal{E}=0} \tag{B10}$$

Continuing for all components of $\boldsymbol{\varepsilon}$ on both V_λ and V_ϕ provides

$$\begin{bmatrix} V'_{\lambda\mathcal{E}} \\ V'_{\phi\mathcal{E}} \end{bmatrix} = \begin{bmatrix} S'_\lambda \\ S'_\phi \end{bmatrix}_{\mathcal{E}=0}\Phi^{-1} \tag{B11}$$

Substituting (B11) into (B9) and observing from Eqs. (11) and (12) that

$$J\bigg|_{\mathcal{E}=0} = \begin{bmatrix} S'_\lambda \\ S'_\phi \end{bmatrix}\Phi^{-1}[S_\lambda S_\phi]$$

provides

$$E\begin{bmatrix} \Delta\lambda^2 & \Delta\lambda\Delta\phi \\ \Delta\lambda\Delta\phi & \Delta\phi^2 \end{bmatrix} = J^{-1}\bigg|_{\mathcal{E}=0} \tag{B12}$$

for sufficiently small noise. The assertion of this section is, therefore, verified.

References

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